

The X-ray transform on 2-step nilpotent Lie groups of higher rank

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Dedicated to the memory of Sergio Console.

Abstract

We prove injectivity and a support theorem for the X-ray transform on 2-step nilpotent Lie groups with many totally geodesic 2-dimensional flats. The result follows from a general reduction principle for manifolds with uniformly escaping geodesics.

1 Background

The X-ray transform of a sufficiently rapidly decreasing continuous function f on the Euclidean plane \mathbb{R}^2 is a function $\mathcal{X}f$ defined on the set of all straight lines via integration along these lines. More precisely, if ξ is a straight line, given by a point $x \in \xi$ and a unit vector $\theta \in \mathbb{R}^2$ such that $\xi = x + \mathbb{R}\theta$, then

$$\mathcal{X}f(\xi) = \mathcal{X}f(x, \theta) = \int_{-\infty}^{\infty} f(x + s\theta) ds.$$

It is natural to ask about injectivity of this transform and, if yes, for an explicit inversion formula. If $f(x) = O(|x|^{-(2+\epsilon)})$ for some $\epsilon > 0$, the function f can be recovered via the following inversion formula, going back to J. Radon [18] in 1917:

$$(1) \quad f(x) = -\frac{1}{\pi} \int_0^\infty \frac{F'_x(t)}{t} dt,$$

where $F_x(t)$ is the mean value of $\mathcal{X}f(\xi)$ over all lines ξ at distance t from x :

$$F_x(t) = \frac{1}{2\pi} \int_{S^1} \mathcal{X}f(x + t\theta^\perp, \theta) d\theta,$$

where $(x, y)^\perp = (y, -x)$. Zalcman [29] gave an example of a non-trivial function $f \in C^\infty(\mathbb{R}^2)$ with $f(x) = O(|x|^{-2})$ and $\mathcal{X}f(\xi) = 0$ for all lines $\xi \subset \mathbb{R}^2$ and, therefore, the decay condition for the inversion formula is optimal.

Under stronger decay conditions, it is possible to prove the following support theorem (see [5, Thm. 2.1] or [7, Thm. I.2.6]):

Theorem 1.1 (Support Theorem). *Let $R > 0$ and $f \in C(\mathbb{R}^2)$ with $f(x) = O(|x|^{-k})$ for all $k \in \mathbb{N}$. Assume that $\mathcal{X}f(\xi) = 0$ for all lines ξ with $d(\xi, 0) > R$. Then we have $f(x) = 0$ for all $|x| > R$.*

Again, the stronger decay condition is needed here by a counterexample of D.J. Newman given in Weiss [26] (see also [7, Rmk. I.2.9]). The Euclidean X-ray transform plays a prominent role in medical imaging techniques like the CT and PET (see, e.g., [12]).

The X-ray transform can naturally be generalized to other complete, simply connected Riemannian manifolds, by replacing straight lines by complete geodesics. Radon mentioned in [18] that there is an analogous inversion formula in the (real) hyperbolic plane \mathbb{H}^2 , where the denominator in the integral of (1) has to be replaced by $\sinh(t)$ (see also [7, Thm. III.1.12(ii)]). There is also an analogue of the support theorem for the hyperbolic space (see [7, Thm. III.1.6]), valid for functions f satisfying $f(x) = O(e^{-kd(x_0, x)})$ for all $k \in \mathbb{N}$ and $x_0 \in \mathbb{H}^n$.

In the case of a continuous function f on a *closed* Riemannian manifold X , the domain of $\mathcal{X}f$ is the set of all *closed* geodesics. Continuous functions f can only be recovered from their X-ray transform $\mathcal{X}f$ if the union of all closed geodesics is dense in X . But this condition is not sufficient as the following simple example of the two-sphere \mathbb{S}^2 shows. Every *even* continuous function f on \mathbb{S}^2 (i.e., $f(-x) = f(x)$) can be recovered by its integrals over all great circles. This fact and a solution similar to (1) goes back to Minkowski 1911 and Funk 1913 (see [7, Section II.4.A] and the references therein). But, on the other hand, it is easy to see that $\mathcal{X}f$ vanishes for all *odd* functions, so the restriction to even functions is essential. For injectivity and support theorems of the X-ray transform on compact symmetric spaces X other than \mathbb{S}^n see, e.g., [7, Section IV.1]. Injectivity properties of the extended X-ray transform for symmetric k -tensors on closed manifolds (with respect to the solenoidal part) play an important role in connection with *spectral rigidity* (see [4]) and were proved for closed manifolds with Anosov geodesic flows (see [3, Thms 1.1 and 1.3] for $k = 0, 1$) or strictly negative curvature (see [2] for arbitrary $k \in \mathbb{N}$).

Another class of manifolds for which the X-ray transform and its extension to symmetric k -tensors has been studied are *simple manifolds*, i.e., manifolds X with strictly convex boundary and without conjugate points (see [23]). An application is the *boundary rigidity problem*, i.e., whether it is possible to reconstruct the metric of X (modulo isometries fixing the boundary) from the knowledge of the distance function between points on the boundary ∂X . Solenoidal injectivity is known for $k = 0, 1$ for all simple manifolds (see [13] and [1]), and for all $k \in \mathbb{N}$ for surfaces [16] and for negatively curved manifolds [15]. There are also support type theorem for the X-ray transform on simple manifolds (see [10, 11] and [25] and the references therein). A very recommendable survey with a list of open problems is [17].

2 A reduction principle for manifolds with uniformly escaping geodesics

In this article, we will only consider complete Riemannian manifolds X whose geodesics escape in the sense of e.g. [27], [28], [9], in a uniform way. Simply connected manifolds without conjugate points have this property, but we like to stress that the main examples in this article will be *manifolds with conjugate points*. Geodesics will always be parametrized by arc length.

Definition 2.1. *A Riemannian manifold X has uniformly escaping geodesics if for each $r \in \mathbb{R}_0^+$ there is $P(r) \in \mathbb{R}_0^+$ such that for every geodesic $\gamma: \mathbb{R} \rightarrow X$ and every $t > P(r)$, we have $d(\gamma(t), \gamma(0)) > r$. We call P an escape function of X .*

The smallest such function P ,

$$P(r) := \sup\{t \geq 0 \mid \exists \text{ geodesic } \gamma: \mathbb{R} \rightarrow X, d(\gamma(0), \gamma(t)) \leq r\}$$

is thus required to be finite for all r . After time $P(r)$ every geodesic has left a closed ball $B_r(p)$ of radius $r \in \mathbb{R}_0^+$ around its center $p \in X$. The function P increases and satisfies $P(r) \geq r$. Note that P may not be continuous.

Manifolds with this property must be simply connected and non-compact. As mentioned earlier, simply connected Riemannian manifolds without conjugate points have this property with escape function $P(r) = r$.

The class of compactly supported continuous functions on such a manifold is preserved under restriction to totally geodesic immersed submanifolds. Thus if f is a compactly supported continuous function on X , say $\text{supp}(f) \subset B_r(p)$ for some $p \in X$ and $r > 0$, and $\phi: Y \rightarrow X$ a totally geodesic isometric immersion, then f has compact support on Y and $\text{supp}(f \circ \phi) \subset B_{P(r)}^Y(p)$. In particular, this holds for geodesics (as 1-dimensional immersions) and the integral of f over any geodesic in X is thus defined.

Before we formulate the reduction principle, let us first fix some notation. The unit tangent bundle of X is denoted by SX . For a Riemannian manifold X let $C_c(X)$ be the space of all continuous functions $f: X \rightarrow \mathbb{C}$ with compact support. By $G(X)$ we denote the set of (unparametrized oriented) geodesics, i.e.

$$G(X) = \{\gamma(\mathbb{R}) \mid \gamma: \mathbb{R} \rightarrow X \text{ geodesic}\}$$

The X-ray transform of $f \in C_c(X)$ is the function $\mathcal{X}f: G(X) \rightarrow \mathbb{C}$ with

$$\mathcal{X}f(L) = \int_L f = \int_{-\infty}^{+\infty} f(\gamma(t)) dt$$

if $L = \gamma(\mathbb{R})$ and γ a unit speed geodesic.

Definition 2.2. *Let $r_0 \geq 0$ and $\sigma: [r_0, \infty) \rightarrow \mathbb{R}_0^+$ be a function. We say that the σ -support theorem holds on X if for $p \in X$ and $f \in C_c(X)$, $r \in [r_0, \infty)$ we have that $\mathcal{X}f|_{G(X \setminus B_{\sigma(r)}(p))} = 0$ implies $f|_{X \setminus B_r(p)} = 0$. We say that X has a support theorem if this holds for a function σ with $\lim_{r \rightarrow \infty} \sigma(r) = \infty$.*

Remark 2.3. If X has a σ -support theorem, then X has a support theorem for all smaller functions as well. Moreover, we can always modify $\sigma : [r_0, \infty) \rightarrow \mathbb{R}_0^+$ to be monotone non-decreasing. If $r_0 = 0$, i.e., $\sigma : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, the σ -support theorem implies injectivity of the X -ray transform.

Then we have the following reduction principle.

Theorem 2.4. Let X be a complete, Riemannian manifold which has uniformly escaping geodesics with escape function P .

- (i) Assume there exists, for every $x \in X$, a closed totally geodesic immersed submanifold $Y \subset X$ through x such that the X -ray transform on Y is injective. Then the X -ray transform on X is also injective.
- (ii) Let $\mu : [r_0, \infty) \rightarrow \mathbb{R}_0^+$ be a function with $\mu \geq P(0)$. Assume there exists, for every $v \in SX$, a closed totally geodesic immersed submanifold $Y \subset X$ with $v \in SY$ such that the μ -support theorem holds on Y . Then a σ -support theorem holds on X for any function $\sigma : [r_0, \infty) \rightarrow \mathbb{R}_0^+$ with $P(\sigma(r)) \leq \mu(r)$ for all $r \geq r_0$. In particular, we can choose σ to be unbounded if μ is unbounded.

Proof. (i) is obviously true by restriction since all geodesics in Y are also geodesics in X .

For (ii), let $f \in C_c(X)$ and $r \geq r_0$. We fix a point $p \in X$ and let \mathcal{Y}_p be a set of closed totally geodesic immersed submanifolds Y with μ -support theorem and so that each geodesic through p lies in one of the $Y \in \mathcal{Y}_p$.

We then have

$$f|_{X \setminus B_r^X(p)} = 0$$

if

$$\forall Y \in \mathcal{Y}_p : f|_{Y \setminus B_r^Y(p)} = 0,$$

since, by assumption, each geodesic in X is contained in some Y . Now, by the μ -support theorem in $Y \in \mathcal{Y}_p$, we have

$$f|_{Y \setminus B_r^Y(p)} = 0$$

if

$$\mathcal{X}f|_{G(Y \setminus B_{\mu(r)}^Y(p))} = 0.$$

Since X has uniformly escaping geodesics property, this is guaranteed if

$$\mathcal{X}f|_{G(X \setminus B_s^X(p))} = 0$$

for any $s \geq 0$ with $P(s) \leq \mu(r)$. Thus X has a σ -support theorem for any function $\sigma : [r_0, \infty) \rightarrow \mathbb{R}_0^+$ satisfying $P(\sigma(r)) \leq \mu(r)$. \square

Remark 2.5. If the escape function $P : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is left-continuous, i.e. $\lim_{s \nearrow r} P(s) = P(r)$, we can choose $\sigma(r) = \sup\{s \geq 0 \mid P(s) \leq \mu(r)\}$.

3 Applications of the reduction principle

In this section we demonstrate that many interesting examples can be derived by the reduction principle from \mathbb{R}^2 and \mathbb{H}^2 . The X-ray transform on the euclidean and on the hyperbolic plane is injective and both have a μ -support theorem with $\mu(r) = r$. This follows directly from the euclidean or hyperbolic version of Radon's classical inversion formula (1), or Theorem 1.1.

If $X = X_1 \times X_2$ is the product of two Riemannian manifolds of positive dimension with uniformly escaping geodesics, with escape functions P_1 and P_2 respectively, then X has uniformly escaping geodesics with function P satisfying

$$\begin{aligned} \max\{P_1(r), P_2(r)\} &\leq P(r) = \sup \left\{ \sqrt{P_1(r_1)^2 + P_2(r_2)^2} \mid r_1^2 + r_2^2 = r^2 \right\} \\ &\leq P_1(r) + P_2(r). \end{aligned}$$

Each vector $v \in S(X_1 \times X_2)$ lies in a 2-flat $F \subset X_1 \times X_2$, i.e. a totally geodesic immersed flat submanifold. By the reduction principle, the σ -support theorem holds on $X_1 \times X_2$ for any function σ with $P(\sigma(r)) \leq r$ for all $r \in [P(0), \infty)$. Note that this result does *not* require that there are support theorems for the X-ray transforms on the factors X_1 and X_2 .

The reduction principle can also be applied to symmetric spaces of noncompact type. These spaces have no conjugate points and each of their geodesics is contained in a flat of dimension at least 2 if their rank is at least 2. In non-compact rank-1 symmetric spaces each geodesic is contained in a real hyperbolic plane. Therefore, the reduction principle yields injectivity of the X-ray transform and a support theorem with $\sigma(r) = r$ ([6], also [7, Cor. IV.2.1]).

Another interesting family are noncompact harmonic manifolds, which do not have conjugate points. Prominent examples in this family are Damek-Ricci spaces. In [21], Rouviere used the fact that each geodesic of a Damek-Ricci space is contained in a totally geodesic complex hyperbolic plane $\mathbb{C}\mathbb{H}^2$ to obtain a support theorem with $\sigma(r) = r$ for Damek Ricci spaces.

The main result in this article is about injectivity of the X-ray transform and a support theorem for a certain class of 2-step nilpotent Lie groups with a left invariant metric and higher rank introduced in [22]. By [14] these spaces have conjugate points. Therefore, the methods of [10] do not immediately apply to these spaces. The spaces in [22] differ also significantly from Heisenberg-type groups which do not even infinitesimally have higher rank.

3.1 2-step nilpotent Lie groups have uniformly escaping geodesics.

The Lie algebra of a 2-step nilpotent Lie algebra \mathfrak{n} splits orthogonally as $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{z}$, $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$ the commutator and $\mathfrak{h} = \mathfrak{z}^\perp$ its orthogonal complement. We

can thus view $\mathfrak{z} \subset \mathfrak{so}(\mathfrak{h})$ as a vectorspace of skew symmetric endomorphisms of \mathfrak{h} . We have

$$\langle [h, k] \mid z \rangle = \langle zh \mid k \rangle$$

for $h, k \in \mathfrak{h}$, $z \in \mathfrak{z}$. We show that 2-step nilpotent Lie groups have uniformly escaping geodesics, hence the X-ray transform for all functions with compact support is defined.

Theorem 3.1. *Let N be a simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{h}$, $\mathfrak{z} \subset \mathfrak{so}(\mathfrak{h})$. Then N has uniformly escaping geodesics with a continuous escape function P .*

Proof. We will prove that for each $r \in \mathbb{R}_0^+$ there is $P(r) \in \mathbb{R}^+$ such that every geodesic γ with $\gamma(0) = e$ (the neutral element of N) we have that $d(\gamma(t), e) \leq r$ implies $t \leq P(r)$.

We denote by $\exp^n: \mathfrak{n} \rightarrow N$ the exponential map of the Lie group. Since N is simply connected nilpotent this is a diffeomorphism. In particular, $(\exp^n)^{-1}(B_r(e)) \subset B_{\rho(r)}^{\mathfrak{n}}(0)$ for some increasing continuous function $\rho(r): \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ with $\rho(0) = 0$. We will show that there is $P(r)$ such that for every geodesic γ in N with $\gamma(0) = e$, the curve $(\exp^n)^{-1} \circ \gamma$ has left $B_{\rho(r)}^{\mathfrak{n}}(0)$ after time $P(r)$.

From [8] for a geodesic $\gamma(t) = \exp^n(z(t) + h(t))$ with $z(t) \in \mathfrak{z}$, $h(t) \in \mathfrak{h}$, $\gamma'(0) = z_0 + h_0$, we have

$$\begin{aligned} h''(t) &= z_0 h'(t), \\ z'(t) &= z_0 + \frac{1}{2}[h(t), h'(t)], \end{aligned}$$

which we need to solve subject to the initial conditions

$$\gamma(0) = \exp^n(z(0) + h(0)) = e \text{ hence } z(0) = 0 = h(0),$$

$$\gamma'(0) = z_0 + h_0 = z'(0) + h'(0),$$

so that $\|z_0\|^2 + \|h_0\|^2 = 1$. The solution to the first equation is

$$h(t) = ((e^{tz_0} - 1)z_0^{-1}) h_0.$$

Note that this is well defined even if z_0 is not invertible. Inserting this into the second equation gives

$$z'(t) = z_0 + \frac{1}{2} [((e^{tz_0} - 1)z_0^{-1}) h_0, e^{tz_0} h_0].$$

Taking the scalar product of this with z_0 gives

$$\begin{aligned} \langle z'(t) \mid z_0 \rangle &= \|z_0\|^2 + \frac{1}{2} \langle z_0 \mid [((e^{tz_0} - 1)z_0^{-1}) h_0, e^{tz_0} h_0] \rangle \\ &= \|z_0\|^2 + \frac{1}{2} \langle z_0(e^{tz_0} - 1)z_0^{-1} h_0 \mid e^{tz_0} h_0 \rangle \\ &= \|z_0\|^2 + \frac{1}{2} \|h_0\|^2 - \frac{1}{2} \langle h_0 \mid e^{tz_0} h_0 \rangle, \end{aligned}$$

since e^{tz_0} is orthogonal. In order to compute $\langle z(t) | z_0 \rangle$, we integrate,

$$\langle z(t) | z_0 \rangle = t\|z_0\|^2 + \frac{t}{2}\|h_0\|^2 + \frac{1}{2}\langle h_0 | (1 - e^{tz_0})z_0^{-1}h_0 \rangle.$$

It follows that

$$z(t) = tz_0 + \frac{t\|h_0\|^2 + \langle h_0 | (1 - e^{tz_0})z_0^{-1}h_0 \rangle}{2\|z_0\|^2}z_0 + w(t)$$

with $w(t) \in \mathfrak{z}$ perpendicular to z_0 . Hence, in the norm $\|\cdot\|$ of \mathfrak{n} , we can estimate

$$\begin{aligned} \|z(t) + h(t)\|^2 &\geq \|((e^{tz_0} - 1)z_0^{-1})h_0\|^2 + \\ &\quad + \frac{1}{4\|z_0\|^2} (2\|z_0\|^2t + t\|h_0\|^2 + \langle h_0 | (1 - e^{tz_0})z_0^{-1}h_0 \rangle)^2. \end{aligned}$$

We split $\mathfrak{h} = \oplus_{\lambda \in \mathbb{R}} E(z_0, i\lambda)$ into the eigenspaces of z_0 and let $h_{\max} \in E(z_0, i\lambda) = \mathbb{C}$ be the largest component of h_0 , $i\lambda$ the corresponding eigenvalue. Thus $|h_{\max}|^2 \geq \frac{1}{\dim \mathfrak{h}}\|h_0\|^2$. Disregarding all other components, we estimate

$$\begin{aligned} \|z(t) + h(t)\|^2 &\geq \\ &\geq \left| \frac{e^{it\lambda} - 1}{i\lambda} \right|^2 |h_{\max}|^2 + \frac{1}{4\|z_0\|^2} \left(2\|z_0\|^2t + t\|h_{\max}\|^2 + \operatorname{Re} \left(\frac{1 - e^{it\lambda}}{i\lambda} \right) |h_{\max}|^2 \right)^2 \\ &= \frac{2 - 2\cos(\lambda t)}{\lambda^2} |h_{\max}|^2 + \frac{1}{4\|z_0\|^2} \left(2\|z_0\|^2t + \left(t - \frac{\sin(\lambda t)}{\lambda} \right) |h_{\max}|^2 \right)^2 \\ &= \|z_0\|^2t^2 + |h_{\max}|^2 \left(\frac{2 - 2\cos(\lambda t)}{\lambda^2} + t \left(t - \frac{\sin(\lambda t)}{\lambda} \right) + \frac{|h_{\max}|^2}{4\|z_0\|^2} \left(t - \frac{\sin(\lambda t)}{\lambda} \right)^2 \right). \end{aligned}$$

We now consider the cases:

$$\|z_0\|^2 \geq \frac{1}{2}: \text{ Then } \|z(t) + h(t)\|^2 \geq \frac{1}{2}t^2.$$

If $\|z_0\|^2 \leq \frac{1}{2}$, then $\|h_0\|^2 = 1 - \|z_0\|^2 \geq \frac{1}{2}$, hence $|h_{\max}|^2 \geq \frac{1}{2\dim \mathfrak{h}}$. We can therefore estimate

$$\|z(t) + h(t)\|^2 \geq \frac{1}{2\dim \mathfrak{h}} \left(\frac{2 - 2\cos(\lambda t)}{\lambda^2} + t \left(t - \frac{\sin(\lambda t)}{\lambda} \right) + \frac{1}{4\dim \mathfrak{h}} \left(t - \frac{\sin(\lambda t)}{\lambda} \right)^2 \right).$$

If $\lambda = 0$ the bracket evaluates to t^2 , hence $\|z(t) + h(t)\|^2 \geq \frac{1}{2\dim \mathfrak{h}}t^2$.

If $0 \leq t \leq \frac{\pi}{2\lambda}$ then $\cos(\lambda t) \leq 1 - \frac{1}{2}(\lambda t)^2$. The other two summands are always nonnegative. Hence in this case,

$$\|z(t) + h(t)\|^2 \geq \frac{t^2}{2\dim \mathfrak{h}}.$$

If $t > \frac{\pi}{2\lambda}$ then $t - \frac{\sin(\lambda t)}{\lambda} \geq (\frac{\pi}{2} - 1)t$. Observing that the rightmost and the leftmost summand are nonnegative, we get in this case that

$$\|z(t) + h(t)\|^2 \geq \frac{(\pi - 2)t^2}{4\dim \mathfrak{h}}.$$

Thus we have shown that

$$\|z(t) + h(t)\|^2 \geq t^2 \min \left\{ \frac{1}{2}, \frac{1}{2\dim \mathfrak{h}}, \frac{\pi - 2}{4\dim \mathfrak{h}} \right\} = t^2 \frac{\pi - 2}{4\dim \mathfrak{h}}.$$

Thus the curve $(\exp^n)^{-1}(\gamma(t)) = z(t) + h(t)$ has left $B_{\rho(r)}^n(0)$ after time $t = P(r) := \rho(r)\sqrt{\frac{4\dim \mathfrak{h}}{\pi - 2}}$. \square

3.2 X-ray transform on certain 2-step nilpotent Lie groups

Let $\mathfrak{h} = \mathbb{R}^{2q} = \mathbb{C}^q$ and $\mathfrak{z} = \mathfrak{t}_{q-1} \subset \mathfrak{su}(q) \subset \mathfrak{so}(2q)$ be the Lie algebra of the maximal torus of $SU(q)$ and consider the 2-step nilpotent Lie group N_q with Lie algebra $\mathfrak{n}_q = \mathfrak{z} \oplus \mathfrak{h} = \mathfrak{t}_q \oplus \mathbb{R}^{2q}$ endowed with a left invariant metric. In [22], it was shown that for every $q \in \mathbb{N}$, $q \geq 3$, the Lie group N_q has the property that each geodesic is contained in a totally geodesic immersed 2-dimensional flat submanifold. The reduction principle and Theorem 3.1 yield

Theorem 3.2. *The X-ray transform on N_q is injective and has a support theorem.*

Remark 3.3. *Since the escape function P , defined at the end of the proof of Theorem 3.1, is continuous, N_q admits a σ -support theorem with $\sigma(r) = \sup\{s \geq 0 \mid P(s) \leq r\}$, due to Remark 2.5.*

Moreover, the σ -support theorem can be extended to general compact sets (not only metric balls) in N_q . This extension is based on the following direct consequence of the classical support theorem (Theorem 1.1) for the X-ray transform on \mathbb{R}^2 : Let $K_0 \subset \mathbb{R}^2$ be a compact set and $\text{conv}(K_0) \subset \mathbb{R}^2$ be its convex hull. Let $f \in C(\mathbb{R}^2)$ with decay conditions as in Theorem 1.1. Then $\mathcal{X}f|_{G(\mathbb{R}^2 \setminus K_0)} = 0$ implies $f|_{\mathbb{R}^2 \setminus \text{conv}(K_0)} = 0$ (see [7, Cor. I.2.8]). Using this fact, we conclude for any compact set $K \subset N_q$, any point $p \in N_q$, and any $f \in C_c(N_q)$ with $\mathcal{X}f|_{G(N_q \setminus K)} = 0$ that

$$f|_{N_q \setminus \text{conv}_p(K)} = 0,$$

where

$$\text{conv}_p(K) = \{x \in X \mid \forall Y \in \mathcal{Y}_p \text{ with } x \in Y : x \in \text{conv}_Y(K \cap Y)\}$$

and

- \mathcal{Y}_p is a set of totally geodesic immersions of submanifolds isometric to \mathbb{R}^2 such that each geodesic through p lies in one of the $Y \in \mathcal{Y}_p$,

- $\text{conv}_Y(Z)$ denotes the convex hull of a subset Z of $Y \cong \mathbb{R}^2$.

The proof is a straightforward modification of the proof of Theorem 2.4. The σ -support theorem is then just the special case $K = B_{\sigma(r)}(p)$, since then $\text{conv}_p(K) \subset B_r(p)$.

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